# Third-Order Intermodulation Distortion in Cascaded Stages

S. A. Maas, *Fellow, IEEE*

***Abstract*— The relation for intermodulation distortion (IM) levels in cascaded stages is based on the worst-case assumption that intermodulation products at the output of each stage combine in phase. However, because there is no apparent reason why IM products should always combine in phase, efforts are customarily made to discard this assumption. Using Volterra analysis, we show that, in certain cases, the IM products do indeed combine in phase, and the cascade relation gives an accurate, not worst-case, prediction of distortion.**

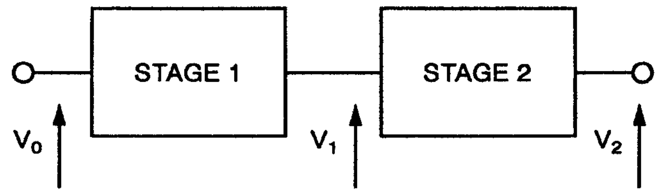


Fig. 1. Cascaded weakly nonlinear stages.

## I. Introduction

THE $n$th-order output intermodulation (IM) intercept point of a cascade of stages (Fig. 1) is given by the following well known expression [1]:

$$\mathrm{IP}_n^{(1-n)/2} = \mathrm{IP}_{n,M}^{(1-n)/2} + (G_M \mathrm{IP}_{n,M-1})^{(1-n)/2} + (G_M G_{M-1} \mathrm{IP}_{n,M-2})^{(1-n)/2} + \cdots \quad (1)$$

where $\mathrm{IP}_{n,m}$ is the $n$th-order intermodulation intercept point of the $m$th stage, and $G_m$ is the gain of the $m$th stage. The summation is carried out over all $M$ stages in the cascade. Each term in (1) represents a contribution to the output intercept point from one stage in the cascade, beginning with the last.

In the derivation of (1), it is assumed that the distortion currents or voltages generated by each stage combine in phase with those generated by previous stages. Since there appears *a priori* to be no reason for a relationship between the phases of these distortion components, this becomes a worst-case assumption. Furthermore, if the phases of the distortion components are indeed random, it is unlikely in long cascades that all IM products will combine in phase; the distortion components should then combine powerwise, and the IM intercept point should be greater than that given by (1). Because of this reasoning, it is common practice in the design of microwave systems to modify (1) so that the terms represent power combination, and higher $\mathrm{IP}_n$ than that given by (1) is predicted.

In this letter we examine the validity of (1) by means of Volterra-series analysis. We show that, under conditions where Volterra analysis is valid (weakly nonlinear or weakly excited circuits), the IM products in certain cases combine in phase or nearly in phase, and the "worst-case" assumption in (1) is valid. In other cases the in-phase assumption is not exact, but may be a much better estimate than often perceived. Because third-order IM is invariably of great concern, this work is limited to third-order IM products.

Manuscript received December 9, 1994.
The author is with Nonlinear Technologies, Inc., Long Beach, CA 90807 USA.
IEEE Log Number 9410994.

## II. Third-Order IM

We consider the cascade of two weakly nonlinear stages shown in Fig. 1. The stages need not be unilateral, but it is necessary that the terminations of each stage be the same as those used to determine the transfer functions.

The distortion component at the output of the first stage is [1], [2]

$$V_1(2\omega_2 - \omega_1) = V_0(\omega_2)V_0(\omega_2)V_0^*(\omega_1)H_{3,1}(\omega_2, \omega_2, -\omega_1). \quad (2)$$

$H_{n,m}$ is the $n$th-order nonlinear transfer function of the $m$th stage [2], and the asterisk signifies the complex conjugate. The distortion output of stage 2 has two components: 1) the distortion generated in stage 2 and 2)the distortion from stage 1, transferred to the output via the linear response

$$V_2(2\omega_2 - \omega_1) = V_1(\omega_2)V_1(\omega_2)V_1^*(\omega_1)H_{3,2}(\omega_2, \omega_2, -\omega_1) + V_1(2\omega_2 - \omega_1)H_{1,2}(2\omega_2 - \omega_1). \quad (3)$$

Noting that

$$V_1(\omega_n) = V_0(\omega_n)H_{1,1}(\omega_n)$$
$$V_2(\omega_n) = V_1(\omega_n)H_{1,2}(\omega_n) \quad (4)$$

and substituting (4) and (2) into (3), we obtain

$$V_2(2\omega_2 - \omega_1) = V_0^2(\omega_2)V_0^*(\omega_1)H_{1,1}^2(\omega_2)H_{1,1}^*(\omega_1)H_{3,2}(\omega_2, \omega_2, -\omega_1) + V_0^2(\omega_2)V_0^*(\omega_1)H_{3,1}(\omega_2, \omega_2, -\omega_1)H_{1,2}(2\omega_2 - \omega_1) \quad (5)$$

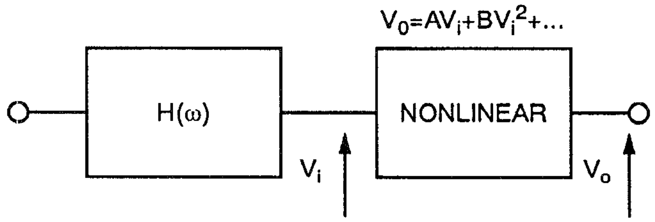


Fig. 2. A single weakly nonlinear stage (e.g., each stage in Fig. 1) often can be modeled as a linear stage followed by a nonlinear one. The nonlinearity is assumed to be memoryless and is characterized by a Taylor series in the vicinity of its dc bias point.

where, again, the first additive term represents the distortion generated in stage 2, and the second represents the distortion generated in stage 1. To determine whether these components combine in phase, we must examine these two terms. Specifically, is there a relationship between the phases of $H_{1,1}^2(\omega_2)H_{1,1}^*(\omega_1)H_{3,2}(\omega_2,\omega_2,-\omega_1)$ and $H_{3,1}(\omega_2,\omega_2,-\omega_1)$ $H_{1,2}(2\omega_2-\omega_1)$? We examine several cases of interest.

### A. Identical Stages

If the stages are identical, $H_{3,2} = H_{3,1}, H_{1,2} = H_{1,1}$ and the phase difference between the two terms is $\angle H_{1,1}^2(\omega_2)H_{1,1}^*(\omega_1) - \angle H_{1,1}(2\omega_2 - \omega_1)$. These approach equality as $\omega_1 \rightarrow \omega_2$ and thus $\omega_1 \approx \omega_2 \approx 2\omega_2 - \omega_1$. As long as the phase of $H_{1,1}(\omega)$ varies slowly with frequency or the difference in frequency between $\omega_1$ and $\omega_2$ is small, the phase difference between these terms will be small. The IM products then combine in phase as $\omega_1 \rightarrow \omega_2$, and (1) is valid. In microwave components (e.g. multistage amplifiers) that consist of identical or nearly identical stages, (1) gives a valid estimate of the two-tone third-order IM intercept point.

### B. Dissimilar Stages

If the stages are dissimilar, at first glance there appears to be no relation between the two terms in (5). However, in certain cases it is possible to make some generalizations.

Many two-port electronic components can be modeled as shown in Fig. 2. This model consists of a linear stage followed by a nonlinear one. (A subsequent linear stage could be added as well; however, such a stage could be included in the linear part of the following stage, so it need not be part of this model.) This model is used extensively in system analysis, and describes many types of components accurately; for example, it is a good model of a FET amplifier. In such components

$$H_{n,m}(\omega_p,\omega_q,\ldots) = C_n H_{1,m}(\omega_p)H_{1,m}(\omega_q)\cdots \tag{6}$$

where $n$ is the order of the nonlinear transfer function and $C_n$ is a real constant [2]. Substituting this relationship into (5) shows that the phase difference again is $\angle H_{1,2}^2(\omega_2)H_{1,2}^*(\omega_1) - \angle H_{1,2}(2\omega_2-\omega_1)$. This is zero under the same conditions as for identical stages, i.e., as $\omega_1 \rightarrow \omega_2$.

### C. Intervening Linear Stages

Even if the cascaded stages are identical, they are often connected by intervening linear stages: a transmission line, a filter, etc. Obviously, in dissimilar stages, the linear stage could be included in the linear part of the second stage, so all the conclusions for the cascade of dissimilar stages are valid.

Although any valid conclusions about dissimilar stages should also be true of identical stages, our conclusions for dissimilar stages were based on a special model (Fig. 2) that is not always valid. It is possible to be more general. Using the same approach as in the derivation of (5), we find that the analogous relation is

$$\begin{aligned} V_2(2\omega_2-\omega_1) &= V_0^2(\omega_2)V_0^*(\omega_1)H_{1,1}^2(\omega_2)H_{1,1}^*(\omega_1) \\ &\quad \times H_{3,2}(\omega_2,\omega_2,-\omega_1)H_L^2(\omega_2)H_L^*(\omega_1) \\ &\quad + V_0^2(\omega_2)V_0^*(\omega_1)H_{3,1}(\omega_2,\omega_2,-\omega_1) \\ &\quad \times H_{1,2}(2\omega_2-\omega_1)H_L(2\omega_2-\omega_1) \end{aligned} \tag{7}$$

where $H_L$ is the linear transfer function of the intervening linear stage. The conclusions for identical stages are valid here if $\angle H_L^2(\omega_2)H_L^*(\omega_1) = \angle H_L(2\omega_2-\omega_1)$. This equality becomes exact as $\omega_1 \rightarrow \omega_2$. However, these terms show that the linear stage provides additional phase shift, and increases the phase variation with frequency. The greater the phase shift in the linear element, the closer in frequency $\omega_2$ and $\omega_1$ must be for (1) to be valid. Also, if the linear stage introduces a large phase shift at one frequency but not the other, as would happen if the linear stage were a filter and one frequency were close to its band edge, (1) might not be valid.

### D. Third Harmonics

For third harmonics of a single excitation $\omega$, the intermodulation product of interest is $3\omega = \omega+\omega+\omega$. By inspection, (5) becomes

$$\begin{aligned} V_2(3\omega) &= V_0^3(\omega)H_{1,1}^3(\omega)H_{3,2}(\omega,\omega,\omega) \\ &\quad + V_0^3(\omega)H_{3,1}(\omega,\omega,\omega)H_{1,2}(3\omega). \end{aligned} \tag{8}$$

For identical stages the critical question is whether $\angle H_1^3(\omega) = \angle H_1(3\omega)$. This will be true, and (1) will be valid, only if $\angle H_1(\omega) \propto \omega$ at all frequencies up to $3\omega$. Although it is not impossible in some cases for this condition to be met, most broadband microwave systems do not exhibit this characteristic over such a broad frequency range.

We note that the same criteria are necessary for dissimilar stages or for stages separated by a linear element. This will not be proven here, but can be proven by the same techniques.

## III. Discussion and Conclusion

In a system consisting of a chain of identical components or dissimilar components that can be modeled as shown in Fig. 2, the cascade expression (1) is an accurate predictor of two-tone third-order IM intercept point. It is valid in these cases when there are two excitation tones, closely spaced in frequency. There are many systems that meet these conditions; for example, an amplifier consisting of a cascade of identical stages usually meets them.

An invariant requirement of the validity of (1) in two-tone excitation is that $\omega_1 \approx \omega_2$. This is usually the case in two-tone IM tests; however, in operational systems, excitations are not necessarily close in frequency. In these, however, the IM level will be equal to or less than that predicted by (1), and thus (1) still stands as a reasonable worst-case estimate. The deviation from (1) as $|\omega_1 - \omega_2|$ increases is usually very gradual; thus, even systems not strictly meeting the necessary conditions often will be better characterized by (1) than by combining IM powers instead of voltages.

These results have other practical implications. For example, they imply that the conventional two-tone test used to measure the $IP_3$ of a quasilinear amplifier usually measures the worst-case IM level. This result is important in characterizing microwave systems. Also, system simulators using the model of Fig. 2 to characterize nonlinear stages will always predict the $IP_3$ given by (1).

## REFERENCES

[1] S. A. Maas, *Nonlinear Microwave Circuits.* Norwood, MA: Artech House, 1988.

[2] D. D. Weiner and J. F. Spina, *Sinusoidal Analysis and Modeling of Weakly Nonlinear Circuits.* New York: Van Nostrand Reinhold, 1980.